Valuation of the surrender option in unit-linked life insurance policies in a non-rational behaviour framework

Luca Anzilli e Luigi De Cesare

Quaderno n. 20/2007

“Esemplare fuori commercio per il deposito legale agli effetti della legge 15 aprile 2004 n. 106”

Quaderno riprodotto al Dipartimento di Scienze Economiche, Matematiche e Statistiche nel mese di ottobre 2007 e depositato ai sensi di legge

Authors only are responsible for the content of this preprint.
Valuation of the surrender option in unit-linked life insurance policies in a non-rational behaviour framework

Luca Anzilli

Dipartimento di Scienze Economiche e Matematico-Statistiche, Università del Salento. Complesso Ecotekne, Via per Monteroni - 73100 Lecce (Italy)

Luigi De Cesare

Dipartimento di Scienze Economiche, Matematiche e Statistiche, Università di Foggia. Largo Papa Giovanni Paolo II - 71100 Foggia (Italy)

Abstract

In this article we propose a discrete time-based model for the evaluation of the surrender option implicit in a portfolio of single premium unit-linked life policies. We presume that the policyholders do not act rationally. Their behaviour is linked to the credibility of the insurance companies, to the analysis of the economic convenience of a rating agency, as well as to their personal needs for surrenders. In this paper we investigate the effects of a company’s advertising campaign on the price of surrender options. The model was numerically implemented using the Cox et al. [Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option Pricing: A Simplified Approach. Journal of Financial Economics 7, 229-263] binomial model.

Key words: Surrender option, Unit-linked life insurance, Non-rational behaviour
JEL: C61, G13, G22
MSC: IE10, IE50, IB10

1 Introduction

The subject of this study is the evaluation of surrender options implicit in a portfolio of policies of a life insurance company. In these policies, the policy-
holder has the right to withdraw from the contract before its expiry, receiving as a consequence a sum defined *surrender value*.

It is well-known that the determination of the surrender option premium (and more generally, options that are implicit in an insurance policy) is important in analysing the premiums and the reserves, as well as in defining the appropriate coverage to be implemented, and in evaluating the entire insurance company.

In literature, different models have been proposed for the evaluation of the surrender option. Grosen and Jørgensen (1997, 2000), Tanskanen and Lukkarinen (2003), Bacinello (2005) and Shen and Xu (2005) assume that policyholders optimally exercise their surrender options. The surrender decision is the consequence of a rational choice (optimal behaviour). The surrender option is treated as a financial option of the American type and consequently, all the policyholders surrender when the surrender value is greater than the continuity value. This approach pursues the objective of fair valuation of liabilities of the insurance company.

Following Albizzati and Geman (1994), we start from the general consideration that the majority of policyholders is not able to carry out an analysis of economic convenience regarding the opportunity of surrendering the insurance policy.

We hypothesize, instead, that an external agency may carry out the analysis of economic convenience in the surrender operation and then spread the results of such an analysis by means of a specialized press or other means of communication.

Such a message will be perceived in different ways by policyholders who may decide whether to cash in the policy, considering other factors, as for example their trust in the company. The company therefore can carry out an advertising campaign to increase its credibility towards its investors. In this way, the company may discourage surrenders.

Besides, we presume that a part of the surrenders could be motivated by the policyholder’s strictly personal circumstances (as for example an urgent need for cash) which do not depend on the trend of financial markets.

In order to consider these aspects, we introduce a probability of surrender that depends on a function which measures the effectiveness of the message regarding the economic convenience of the surrender option. The advertising campaign has the effect of modifying this function. The case of an optimal rational behaviour of all policyholders is evaluated as a special case.

In this study, we intend to propose a discrete time-based model for the evaluation of a surrender option in a portfolio of unit-linked policies. Some numerical results are shown, illustrating the effects of a company’s advertising campaign on the price of surrender options.

The analysis of the results reveals that, all other conditions being equal, the company’s advertising investment in some cases causes an increase in the surrender option price and in other cases it causes a decrease. These considerations may be useful for company management in order to define appropriate
advertising strategies.
The present paper is organized as follows. In the next section we describe
the structure of the contract. Section 3 contains the valuation framework. In
Section 4 we apply our model for pricing the surrender options embedded in
a single premium unit-linked endowment portfolio with minimum guarantees.
Section 5 contains some concluding remarks.

2 The structure of the contract

In this paper we consider a portfolio of identical unit-linked life insurance
endowment policies with surrender option, issued at time 0 and expiring at
time $T$ where the time is measured in years and $T$ is an integer.
Under these contracts, the insurance company is obliged to pay a specified
benefit to the beneficiary if the insured dies within the term of the contract or
survives the maturity date moreover the policyholder can close the contract
before the term and convert the future payments into an immediate payment
of a surrender value.
Each policy is paid by a single premium, denoted by $U$, and a part of $U$,
denoted by $K$, is immediately invested in a traded mutual fund which does
not pay any dividend. Benefits are measured in units of the reference fund.
The value of this investment at time $\tau$ is given by

$$F_\tau = N S_\tau,$$

where $S_\tau$ is the unit price of the reference fund at time $\tau$ and $N = K/S_0$ is
the number of units acquired at time 0.
We assume that the rate of return on risk-free assets, denoted by $r$, is deter-
mind and constant.
We discretize every unit period into $M$ subintervals of equal length $h = 1/M$.
We denote by $C_t$, $t = 1, 2, \ldots, MT$, the benefit paid at time $t h$ in case of
death of the insured between times $(t - 1) h$ and $t h$, and by $C^V_{MT}$ the benefit
paid at maturity $T$ if the insured is still alive. We assume that the benefits $C_t$
and $C^V_{MT}$ are given by

$$C_t = f_t(S_{th}) \quad t = 1, 2, \ldots, MT$$
$$C^V_{MT} = f^V_{MT}(S_T)$$

where $f_t$ and $f^V_{MT}$ are suitable functions.
We suppose that the policyholder can surrender the contract at the beginning
of each period. In this case the surrender value at time $t h$ is given by

$$H_t = h_t(S_{th}) \quad t = 1, 2, \ldots, MT - 1$$
where $h_t$ are suitable functions. We consider a portfolio of $A_0$ identical policies such that each insured is aged $x$ at the beginning of the contract. Let $A_t$ and $B_t$ be the number of policies still in force at time $t h$ and the number of insured died between times $(t - 1)h$ and $th$ (i.e. between ages $x + (t - 1)h$ and $x + th$), respectively. We denote by $R_t$, $t = 1, 2, \ldots, MT - 1$, the number of surrenders at time $t h$. Moreover we set $R_0 = 0$. Hence we have

$$ A_t = A_{t-1} - R_{t-1} - B_t. $$

The number of policies still in force after surrender option exercise is given by

$$ \overline{A}_t := A_t - R_t, \quad t = 1, 2, \ldots, MT - 1. $$

The liabilities of the insurer paid for deaths, for early exercises and for survivals at maturity date can be represented respectively as

$$ D_t = B_t \, C_t \quad t = 1, 2, \ldots, MT \quad \text{payable at time } t h, $$

$$ E_t = R_t \, H_t \quad t = 1, 2, \ldots, MT - 1 \quad \text{payable at time } t h, $$

$$ D^V_{MT} = A_{MT} \, C^V_{MT} \quad \text{payable at time } T. $$

3 Valuation methodology

In this section we compute the value of the surrender option embedded in the contract under the assumption that financial markets are perfectly competitive, frictionless and arbitrage-free. We also assume that all random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$, where $P$ is the natural probability measure. We define the information available at time $t h$ by the filtration

$$ \begin{cases} 
\mathcal{F}_0 = \sigma (A_0, S_0) \\
\mathcal{F}_t = \sigma (A_0; S_0, s_h, \ldots, s_{th}; B_1, B_2, \ldots, B_t; R_1, R_2, \ldots, R_{t-1}) \\
\quad \quad \quad \quad t = 1, 2, \ldots, MT.
\end{cases} $$
We also define the information available at time $t h$ after surrender exercise by

$$\mathcal{F}_t = \sigma (A_0; S_0, S_h, \ldots, S_{th}; B_1, B_2, \ldots, B_t; R_1, R_2, \ldots, R_t) = \mathcal{F}_t \lor \sigma (R_t)$$

$t = 1, 2, \ldots, MT - 1$.

Hence we have

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_t \subset \cdots \subset \mathcal{F}_{MT}.$$

We observe that random variables $C_t, B_t, H_t$ and $A_t$ are $\mathcal{F}_t$-measurable and random variables $R_t$ and $A_t$ are $\mathcal{F}_t$-measurable.

We are interested in assigning a value at time 0, denoted by $V(0; Y_t)$, to the liability $Y_t$ payable at time $t h$. Let $P_{k,t} = V(k; Y_t)$ be the value at time $k h$ of the liability $Y_t$ payable at time $t h \geq k h$. We denote by $\mathcal{P}_{k,t}$ the value immediately after surrender exercise of the liability $Y_t$.

By using an arbitrage argument, the price $\mathcal{P}_{k,t}$ can be represented as (see Duffie (1996), De Felice and Moriconi (2005))

$$\mathcal{P}_{k,t} = \mathbb{E} \left[ \pi_{k,k+1} P_{k+1,t} \mid \mathcal{F}_k \right] \quad k = 0, 1, 2, \ldots, t - 1$$

(1)

where the deflator $\pi_{k,k+1}$ is a strictly positive $\mathcal{F}_{k+1}$-measurable random variable.

By assuming stochastic independence between mortality and financial factors the deflator $\pi_{k,k+1}$ can be expressed as the product of two strictly positive $\mathcal{F}_{k+1}$-measurable random variables

$$\pi_{k,k+1} = \chi_{k,k+1} \varphi_{k,k+1}.$$ 

(2)

In this context $\chi_{k,k+1}$ is the mortality deflator and $\varphi_{k,k+1}$ is the financial deflator.

In order to preserve the price additivity, we introduce, for the survival risk component of the valuation problem, a transformed probability measure defined by $dD = \chi_{k,k+1} dP$. In the literature several probability measures $D$ are proposed (Wang (1996, 2000, 2002)).

We shall assume that, under the adjusted probability measure $D$, the deaths in time interval $[k h, (k + 1)h]$, given the number of surrenders at time $k h$, have conditional binomial distribution with parameters $\left( \overline{A}_k / h q_x + kh \right)$ where the probability $/ h q_x + kh$ that the insured dies between times $k h$ and $(k + 1)h$ is computed using first order mortality tables.

Hence we have

$$\mathbb{E} \left[ \chi_{k,k+1} B_{k+1} \mid \mathcal{F}_k \right] = \mathbb{E}^D \left[ B_{k+1} \mid \mathcal{F}_k \right] = \overline{A}_k / h q_x + kh.$$ 

(3)
For the financial component, using the risk-neutral valuation technique, there exists a unique equivalent probability measure $Q$ such that

$$E \left[ \varphi_{k,k+1} X | \mathcal{F}_k \right] = e^{-rh} E^Q \left[ X | \mathcal{F}_k \right]$$

(4)

for any $\mathcal{F}_{k+1}$-measurable random variable $X$.

Using similar arguments, the early surrender component is given by

$$P_{k,t} = E^R \left[ T_{k,t} | \mathcal{F}_k \right]$$

(5)

where $R$ is the corresponding adjusted probability measure.

We shall assume that random variable $R_k$, given the information at time $kh$, has conditional binomial distribution with parameters $(A_k, q_k^R)$, under the probability $R$, where $q_k^R$ is the probability to surrender at time $kh$.

Hence we have

$$E^R [R_k | \mathcal{F}_k] = A_k q_k^R.$$

We assume that the insurance company is risk neutral with respect to surrender risk.

Let us assume that not all the insured behave rationally, both because of personal reasons, and because they do not possess the financial-mathematical tools to evaluate the real economic convenience whether or not to surrender the insurance policy and finally because of the influence of the insurance company’s persuasive advertising campaigns.

Furthermore, let us assume that the insured have the opportunity to benefit from economic assessments carried out by external agencies.

We model the probability to surrender by

$$q_k^R = 1 - (1 - q_e) h (1 - \gamma(h))^h \quad k = 1, 2, \ldots, MT - 1$$

where $q_e$ is the annual probability to surrender by personal reasons, $\theta_k = \theta_k(S_{kh})$ is a decision variable, evaluated by external rating organization, on the convenience to surrender at time $kh$ and $\gamma(h)$ is the annual probability to surrender by economic convenience decisions.

If $\theta_k > 1$, it is convenient to exercise the surrender option and the greater $\theta_k$ is, the more convenient the surrender action will be. If $\theta_k < 1$, it is not convenient to exercise the surrender option and the smaller $\theta_k$ is, the less convenient the surrender action will be.

$\gamma = \gamma(\theta_k)$ represent the effectiveness of mass-media influence on surrender decision (i.e. policyholders are deterred from surrendering their policies by advertising).

To reflect the mass-media influence on surrender decision, we represent $\gamma$ as a nondecreasing piece-wise linear function of the decision variable $\theta_k$ (see Fig. 1).
Fig. 1. The function $\gamma(\theta_k)$.

$$
\gamma(\theta_k) = \begin{cases} 
\gamma^{\min} & \text{if } 0 \leq \theta_k \leq \theta^{\min} \\
\gamma^{\min} + \frac{\theta^{\max} - \gamma^{\min}}{\theta^{\max} - \theta^{\min}} \left(\theta_k - \theta^{\min}\right) & \text{if } \theta^{\min} < \theta_k \leq \theta^{\max} \\
\gamma^{\max} & \text{if } \theta_k > \theta^{\max}
\end{cases}
$$

with

$$0 \leq \theta^{\min} \leq 1 \leq \theta^{\max}$$

$$0 \leq \gamma^{\min} \leq \gamma^{\max} \leq 1.$$

We have the maximum probability for surrender when $\theta_k > \theta^{\max}$, while the minimum probability is obtained when $\theta_k < \theta^{\min}$.

The effect of the advertising campaign lowers the graph of the function $\gamma$. This can be obtained, for example, by moving $\theta^{\min}$ and $\theta^{\max}$ to the right. On the contrary, a bad company reputation has the opposite effect.

The case of rational behaviour from all investors is obtained through

$$\theta^{\min} = \theta^{\max} = 1, \quad \gamma^{\min} = 0, \quad \gamma^{\max} = 1, \quad q_e = 0.$$

In Fig. 2 we plot the probability to surrender $q^R_k$ for different values of the discretization parameter $M$.

In the next proposition we compute the values of the liabilities of the insurer.

**Proposition 1** Under the above assumptions the values at time 0 of the lia-
Fig. 2. The probability to surrender $q_k^R$ as function of $\theta_k$ with $\gamma_{\text{min}} = 0$, $\gamma_{\text{max}} = 1$, $q_e = 0.05$.

Probabilities $D_t$, $E_t$ and $D_{MT}^V$ are given by

$$V(0; D_t) = A_0 \left( t-1 \right) \frac{h}{h} q_x e^{-rT} E^Q \left[ \prod_{m=1}^{t-1} \left( 1 - q_m^R \right) C_t \bigg| F_0 \right] \quad t = 1, \ldots, MT,$$

$$V(0; E_t) = A_0 \left( t-1 \right) \frac{h}{h} p_x e^{-rT} E^Q \left[ \prod_{m=1}^{t-1} \left( 1 - q_m^R \right) q_t^R H_t \bigg| F_0 \right] \quad t = 1, \ldots, MT-1,$$

$$V(0; D_{MT}^V) = A_0 \left( MT \right) \frac{h}{h} p_x e^{-rT} E^Q \left[ \prod_{m=1}^{MT-1} \left( 1 - q_m^R \right) C_{MT}^V \bigg| F_0 \right],$$

where $\left( t-1 \right) \frac{h}{h} q_x$ is the probability that the insured dies between times $(t-1)h$ and $th$ and $\left( t-1 \right) \frac{h}{h} p_x$ is the probability that the insured is still alive at time $th$.

**Proof.** See Appendix. \( \square \)

Notice that if $A_0$ is big enough the insurance company is risk-neutral with respect to mortality risk (assumption justified by pooling arguments).

The value of the unit-linked endowment policy portfolio at time 0 is given by

$$L_0 = \sum_{t=1}^{MT} V(0; D_t) + \sum_{t=1}^{MT-1} V(0; E_t) + V(0; D_{MT}^V).$$

From Proposition 1, we have
\begin{align*}
L_0 &= \sum_{t=1}^{MT} A_0 \left( (t-1)h/q \right) e^{-r t h} E^Q \sum_{m=1}^{t-1} \left( 1 - q^R_m \right) C_t \mid F_0 \bigg] + \\
&+ \sum_{t=1}^{MT-1} A_0 \left( t h/p \right) e^{-r t h} E^Q \sum_{m=1}^{t-1} \left( 1 - q^R_m \right) q^R_t H_t \bigg] \mid F_0 \bigg] + \\
&+ A_0 \left( T h/p \right) e^{-r T h} E^Q \prod_{m=1}^{MT-1} \left( 1 - q^R_m \right) C_{MT} \bigg] \mid F_0 \bigg] .
\end{align*}

The value of the portfolio without surrender options is given by

\begin{align*}
\hat{L}_0 &= \sum_{t=1}^{MT} A_0 \left( (t-1)h/q \right) e^{-r t h} E^Q \left[ C_t \mid F_0 \right] + \\
&+ A_0 \left( T h/p \right) e^{-r T h} E^Q \left[ C_{MT} \mid F_0 \right] .
\end{align*}

Hence for each contract the value at time 0 of the surrender option \( O_0 \) is obtained by calculating the difference between the single premium \( U = L_0/A_0 \) and the single premium of a policy without surrender option \( \hat{U} = \hat{L}_0/A_0 \), i.e.

\[ O_0 = U - \hat{U}_0 . \]

### 4 Numerical results

We compute the surrender option value for a unit-linked endowment portfolio with minimum guarantees. In this framework the benefit paid at time \( t h \) in case of death is

\[ C_t = \max \left\{ F_{t h}, K e^{g h} \right\} \quad t = 1, 2, \ldots, MT \]

where \( g \geq 0 \) is a minimum interest rate guaranteed and the benefit paid at maturity \( T \), if the insured is still alive, is given by

\[ C_{MT}^V = C_{MT} = \max \left\{ F_T, K e^{g T} \right\} . \]

The surrender value at time \( t h \) is given by

\[ H_t = \beta_t \max \left\{ F_{t h}, K e^{\delta t h} \right\} \quad t = 1, 2, \ldots, MT - 1 . \]

where \( \beta_t \) is a surrender penalty applied to the policyholder benefit.

We represent the stochastic evolution of the unit price of the reference fund using a recombining binomial tree (Cox et al. (1979)).

At time \( t h \), for \( t = 0, 1, 2, \ldots, MT \), the unit price of the reference fund \( S_{t h} \) can take only two possible values \( u S_{(t-1)h} \) and \( d S_{(t-1)h} \) with respective risk-neutral
probabilities  \( q = \frac{e^{rh} - d}{u - d} > 0 \) and \( 1 - q = \frac{u - e^{rh}}{u - d} > 0 \), where \( u = e^{\sigma \sqrt{h}} \), \( d = u^{-1} \) and \( \sigma \) is the volatility parameter. In order to avoid arbitrage, we must have the condition \( d < e^{rh} < u \) (see Hull (2003)). Thus we fix the discretization lag \( h \) such that \( h < \sigma^2/r^2 \).

We obtain at time \( th \), \( t + 1 \) nodes. Thus each node in the binomial tree can be represented by a pair \((t, j)\) for \( j = 0, 1, 2, \ldots, t\).

The dynamics of the unit price is described as follows

\[
\begin{cases}
S_0(0) = S_0 \\
S_{th}(j) = u^j d^{-j} S_0 \\
t = 0, 1, \ldots, MT, \quad j = 0, 1, \ldots, t
\end{cases}
\]

where \( S_{th}(j) \) denotes the unit price at the node \((t, j)\).

Our first step is to compute the decision variable \( \theta_t(j) \) at each node \((t, j)\).

To this aim we observe that the value of the policy at maturity \( T \) is given by

\[
V_{MT}(j) = C_{MT}(j) \quad j = 0, 1, \ldots, MT
\]

where \( C_{MT}(j) \) is the benefit paid at maturity \( T \) when the underlying price is \( S_T(j) \).

Let \( C_t(j) \) and \( H_t(j) \) denote respectively the benefit and the surrender value paid at time \( th \) when the underlying price is \( S_{th}(j) \).

We define at time \( th < T \) the continuation value as

\[
W_t(j) = \frac{h}{q_x + h} e^{-rh} \left[ q C_{t+1}(j + 1) + (1 - q) C_{t+1}(j) \right]
\]

\[
+ \frac{h}{p_x + h} e^{-rh} \left[ q V_{t+1}(j + 1) + (1 - q) V_{t+1}(j) \right]
\]

\[
t = 0, 1, \ldots, MT - 1, \quad j = 0, 1, \ldots, t
\]

and the value of the policy as

\[
\nabla_t(j) = \max\{H_t(j), W_t(j)\} \quad t = 1, 2, \ldots, MT - 1, \quad j = 0, 1, \ldots, t.
\]

We assume as decision variable at time \( th \), the ratio

\[
\theta_t(j) = \frac{H_t(j)}{W_t(j)} \quad t = 1, 2, \ldots, MT - 1, \quad j = 0, 1, \ldots, t.
\]

The next step is to evaluate the surrender option value.

From the assumption \( C_{MT} = C_{MT} \), the value of the portfolio \((9)\) is given by

\[
L_0 = \sum_{t=1}^{MT-1} A_0 e^{-rt} E^Q \left[ \prod_{m=1}^{t-1} \left( 1 - q_m^R \right) \left( (t-1)h q_x C_t + th p_x q_t^R H_t \right) \bigg| F_0 \right] + \\
+ A_0 \tau h p_x e^{-r \tau} E^Q \left[ \prod_{m=1}^{MT-1} \left( 1 - q_m^R \right) C_{MT} \bigg| F_0 \right]}
\]
and the value of the portfolio without surrender options (10) is given by

$$
\hat{L}_0 = \sum_{t=1}^{MT} A_0 \frac{(t-1)h}{h} q_x e^{-rt} \sum_{j=0}^{(t-1)} \binom{t}{j} q^j (1-q)^{t-j} \max \{ N S_{th}(j), K e^{\theta th} \} \\
+ A_0 \tau P_x e^{-rT} \sum_{j=0}^{MT} \binom{MT}{j} q^j (1-q)^{MT-j} \max \{ N S_T(j), K e^{\theta T} \} \\
= \sum_{t=1}^{MT-1} A_0 \frac{(t-1)h}{h} q_x e^{-rt} \sum_{j=0}^{(t-1)} \binom{t}{j} q^j (1-q)^{t-j} \max \{ N S_{th}(j), K e^{\theta th} \} \\
+ A_0 \tau h P_x e^{-rT} \sum_{j=0}^{MT} \binom{MT}{j} q^j (1-q)^{MT-j} \max \{ N S_T(j), K e^{\theta T} \}.
$$

The value $L_0$ can be computed by using a binomial tree with non-recombining nodes.

In order to illustrate the effects of a company’s advertising campaign on the price of surrender option we calculate the price of the surrender option for different values of $\theta_{min}$ and $\theta_{max}$.

We chose the following set of parameters: $S_0 = 100$, $N = 1$, $x = 40$, $r = 0.05$, $\sigma = 0.3$, $\beta_t = 1$, $\gamma_{min} = 0$, $\gamma_{max} = 1$. The contract maturity $T$ is set equal to 15 years. Moreover we fix $M = 2$. We use the life table of the Italian Statistics for Males Mortality in 1991. The mortality probabilities at non-integer ages are computed by linear interpolation. In Figs. 3-4 the single premium $U$ is plotted against different values of $\theta_{min}$ and $\theta_{max}$.

![Fig. 3. Single premium for a policy with $g = 0.02$ and $\delta = 0.02$. In a) $q_e = 0$ and in b) $q_e = 0.01$.](image)

Our sensitivity analysis shows that the surrender option price is not monotonic with respect to the thresholds $\theta_{min}$ and $\theta_{max}$. If we move $\theta_{max}$ (see Table 1) or $\theta_{min}$ (see Table 2) to the right, we low the graph of function $\gamma$ and this corresponds to increase the effectiveness of the advertising. Indeed, you can see this effect looking at the rows of the Tables 1-2.

We observe that the greater value of the surrender option price is obtained
Fig. 4. Single premium for a policy with $g = 0.04$ and $\delta = 0.02$. In a) $q_e = 0$ and in b) $q_e = 0.01$.

Table 1
Values of the Surrender Option $O_0$ with $\theta_{\min} = 0.90$ and $q_e = 0.01$

<table>
<thead>
<tr>
<th>$g$</th>
<th>$\delta$</th>
<th>$\theta_{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>5.3016</td>
</tr>
<tr>
<td>0.00</td>
<td>0.02</td>
<td>10.8071</td>
</tr>
<tr>
<td>0.00</td>
<td>0.04</td>
<td>18.4215</td>
</tr>
<tr>
<td>0.02</td>
<td>0.00</td>
<td>-1.4831</td>
</tr>
<tr>
<td>0.02</td>
<td>0.02</td>
<td>2.1035</td>
</tr>
<tr>
<td>0.02</td>
<td>0.04</td>
<td>9.6451</td>
</tr>
<tr>
<td>0.04</td>
<td>0.00</td>
<td>-5.9302</td>
</tr>
<tr>
<td>0.04</td>
<td>0.02</td>
<td>-5.5634</td>
</tr>
<tr>
<td>0.04</td>
<td>0.04</td>
<td>-3.5788</td>
</tr>
</tbody>
</table>

when $\gamma$ is equal to $\gamma$ defined by

$$\gamma(\theta_k) = \begin{cases} 
0 & \text{if } \theta_k \leq 1 \\
1 & \text{if } \theta_k > 1.
\end{cases}$$

By lowing $\gamma$, we obtain that, if $\theta_k < 1$, $\gamma$ is closer to $\gamma$ otherwise, if $\theta_k > 1$, $\gamma$ is more distant from $\gamma$. We suppose that the non-monotonicity is due to these two conflicting behaviours.
Table 2
Values of the Surrender Option $O_0$ with $\theta^{\text{max}} = 1.10$ and $q_{e} = 0.01$

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\delta$</th>
<th>$\theta^{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.50</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-0.8668</td>
</tr>
<tr>
<td>0.00</td>
<td>0.02</td>
<td>0.6845</td>
</tr>
<tr>
<td>0.00</td>
<td>0.04</td>
<td>2.8519</td>
</tr>
<tr>
<td>0.02</td>
<td>0.00</td>
<td>-9.5696</td>
</tr>
<tr>
<td>0.02</td>
<td>0.02</td>
<td>-8.1525</td>
</tr>
<tr>
<td>0.02</td>
<td>0.04</td>
<td>-5.9934</td>
</tr>
<tr>
<td>0.04</td>
<td>0.00</td>
<td>-22.7618</td>
</tr>
<tr>
<td>0.04</td>
<td>0.02</td>
<td>-21.3384</td>
</tr>
<tr>
<td>0.04</td>
<td>0.04</td>
<td>-19.6953</td>
</tr>
</tbody>
</table>

5 Conclusion

We have proposed a discrete time-based model for the evaluation of the surrender option implicit in a portfolio of single premium unit-linked life policies. We have studied the effects of a company’s advertising campaign on the price of surrender options.

The numerical results have shown that the company’s advertising investment in some cases causes an increase in the surrender option price and in other cases it causes a decrease. These considerations may be useful for company management in order to define appropriate advertising strategies.

6 Appendix.

Proof of Proposition 1.

For $t = 1, 2, \ldots, MT$ we define

$$\Psi(k) = A_k \sum_{r(t-k)h} e^{-r(t-k)h} \mathbb{E}\left[ \prod_{m=k}^{t-1} \left( 1 - q_{Rm} \right) C_t \left| F_k \right. \right]$$

$k = 0, 1, \ldots, t - 1$. (11)

Our goal is to prove that

$$V(0; D_t) = \Psi(0).$$

First of all we prove the following lemmas:
Lemma 1

\[ V(t - 1; D_t) = \Psi(t - 1). \]

**Proof.** By using (5) and (1), we obtain

\[ V(t - 1; D_t) = \mathbb{E}^R \left[ \mathbb{E} \left[ \pi_{t-1,t} D_t \mid \mathcal{F}_{t-1} \right] \mid \mathcal{F}_{t-1} \right]. \quad (12) \]

We observe that, from (2) and by stochastic independence between mortality and financial factors, we have

\[ \mathbb{E} \left[ \pi_{t-1,t} D_t \mid \mathcal{F}_{t-1} \right] = \mathbb{E} \left[ \chi_{t-1,t} \varphi_{t-1,t} B_t C_t \mid \mathcal{F}_{t-1} \right] = \mathbb{E} \left[ \chi_{t-1,t} B_t \mid \mathcal{F}_{t-1} \right] \cdot \mathbb{E} \left[ \varphi_{t-1,t} C_t \mid \mathcal{F}_{t-1} \right]. \]

From (3) we have

\[ \mathbb{E} \left[ \chi_{t-1,t} B_t \mid \mathcal{F}_{t-1} \right] = \mathbb{E}^D \left[ B_t \mid \mathcal{F}_{t-1} \right] = \mathcal{A}_{t-1}/ \mathcal{R} \left( x + (t-1)h \right). \]

From (4) and taking into account that \( C_t \) does not depend on the surrenders we have

\[ \mathbb{E} \left[ \varphi_{t-1,t} C_t \mid \mathcal{F}_{t-1} \right] = e^{-rh} \mathbb{E}^Q \left[ C_t \mid \mathcal{F}_{t-1} \right] = e^{-rh} \mathbb{E}^Q \left[ C_t \mid \mathcal{F}_{t-1} \right]. \]

From (12), by using \( \mathcal{F}_{t-1} \)-measurability of \( \mathbb{E}^Q \left[ e^{-rh} C_t \mid \mathcal{F}_{t-1} \right] \), we have

\[ V(t - 1; D_t) = \mathbb{E}^R \left[ \mathbb{E}^D \left[ B_t \mid \mathcal{F}_{t-1} \right] \cdot \mathbb{E}^Q \left[ e^{-rh} C_t \mid \mathcal{F}_{t-1} \right] \mid \mathcal{F}_{t-1} \right] \]

\[ = \mathbb{E}^R \left[ \mathbb{E}^D \left[ B_t \mid \mathcal{F}_{t-1} \right] \mid \mathcal{F}_{t-1} \right] \cdot \mathbb{E}^Q \left[ e^{-rh} C_t \mid \mathcal{F}_{t-1} \right]. \quad (13) \]

Moreover, by using \( \mathcal{F}_{t-1} \)-measurability of \( \mathcal{A}_{t-1} \), we have

\[ \mathbb{E}^R \left[ \mathbb{E}^D \left[ B_t \mid \mathcal{F}_{t-1} \right] \mid \mathcal{F}_{t-1} \right] = \mathbb{E}^R \left[ \mathcal{A}_{t-1}/ \mathcal{R} \left( x + (t-1)h \right) \mid \mathcal{F}_{t-1} \right] \]

\[ = \mathcal{R} \left( x + (t-1)h \right) \mathbb{E}^R \left[ \mathcal{A}_{t-1} - \mathcal{R} \left| \mathcal{F}_{t-1} \right. \right] \]

\[ = \mathcal{R} \left( x + (t-1)h \right) \left( \mathcal{A}_{t-1} - \mathbb{E}^R \left[ \mathcal{R} \mid \mathcal{F}_{t-1} \right] \right) \]

\[ = \mathcal{A}_{t-1}/ \mathcal{R} \left( x + (t-1)h \right) \left( 1 - q_{t-1}^R \right). \]

From (13) and (11), we obtain

\[ V(t - 1; D_t) = \mathcal{A}_{t-1}/ \mathcal{R} \left( x + (t-1)h \right) e^{-rh} \mathbb{E}^Q \left[ \left( 1 - q_{t-1}^R \right) C_t \mid \mathcal{F}_{t-1} \right] = \Psi(t - 1). \square \]
Lemma 2

\[ V(k+1; D_t) = \Psi(k+1) \implies V(k; D_t) = \Psi(k) \quad k = 0, 1, \ldots, t-2. \]

Proof. We define

\[ C^k_t = e^{-r(t-k)h} E^Q \left[ \prod_{m=k}^{t-1} (1 - q^R_m) C_t \mid \mathcal{F}_k \right] \]

hence from (11) we have

\[ \Psi(k) = A_k^{(t-k-1)h/h} q_x + kh C^k_t. \]

We observe that

\[ C^k_t = e^{-r(t-k)h} E^Q \left[ \prod_{m=k}^{t-1} C_t \mid \mathcal{F}_k \right] \]

\[ = e^{-r(t-k-1)h} E^Q \left[ \prod_{m=k+1}^{t-1} (1 - q^R_m) C_t \mid \mathcal{F}_k \right] \]

\[ = e^{-r(t-k-1)h} C^{k+1}_t. \]

From definition of \( V(k; D_t) \), substituting in equations (1) and (5) and using (2) we obtain

\[ V(k; D_t) = \mathbb{E} \left[ \text{E}[\chi_{k,k+1} \varphi_{k,k+1} \mid \mathcal{F}_k] \mid \mathcal{F}_k \right]. \]

By using the induction hypothesis

\[ V(k+1; D_t) = \Psi(k+1) \]

we have

\[ V(k; D_t) = \mathbb{E} \left[ \text{E}[\chi_{k,k+1} \varphi_{k,k+1} \Psi(k+1) \mid \mathcal{F}_k] \mid \mathcal{F}_k \right]. \]

(14)

From stochastic independence between mortality and financial factors, we have

\[ \mathbb{E} \left[ \chi_{k,k+1} \varphi_{k,k+1} \Psi(k+1) \mid \mathcal{F}_k \right] = \]

\[ = (t-k-2)h/h q_x (k+1)h \mathbb{E} \left[ \chi_{k,k+1} A_{k+1} \mid \mathcal{F}_k \right] \cdot \mathbb{E} \left[ \varphi_{k,k+1} C^{k+1}_t \mid \mathcal{F}_k \right]. \]

(15)
We calculate now the first expected value in the equation (15). From (3), we have

$$E \left[ \chi_{k,k+1} A_{k+1} \mid \mathcal{F}_k \right] = E^D \left[ A_{k+1} \mid \mathcal{F}_k \right] = E^D \left[ A_k - R_k - B_{k+1} \mid \mathcal{F}_k \right]$$

By using (4), the second expected value in the equation (15) is

$$E \left[ \varphi_{k,k+1} C_{t+1}^k \mid \mathcal{F}_k \right] = E^Q \left[ e^{-rh} C_{t+1}^k \mid \mathcal{F}_k \right] = E^Q \left[ e^{-rh} C_{t+1}^k \mid \mathcal{F}_k \right].$$

Substituting in (14) we obtain the following result

$$V(k; D_t) =$$

$$= (t-k-2)h/hq_{x+(k+1)h} E^R \left[ (A_k - R_k) \left( 1 - /h q_{x+kh} \right) \mid \mathcal{F}_k \right] \cdot E^Q \left[ e^{-rh} C_{t+1}^k \mid \mathcal{F}_k \right]$$

Because of $e^{-rh} E^Q \left[ (1 - q^R_k) C_{t+1}^k \mid \mathcal{F}_k \right] = C_t^k$, we obtain

$$V(k; D_t) = A_k (t-k-1)h/hq_{x+kh} C_t^k = \Psi(k).$$

This completes the proof. \( \Box \)

From Lemma 1 and Lemma 2 we obtain

$$V(k; D_t) = \Psi(k) \quad k = 0, 1, \ldots, t - 1.$$  

Then (6) is proved.

In order to prove (7), for $t = 1, 2, \ldots, MT - 1$ we define

$$\Theta(k) = A_k (t-k)h/p_{x+kh} e^{-r(t-k)h} E^Q \left[ \prod_{m=k}^{t-1} \left( 1 - q^R_m \right) q^R_t H_t \mid \mathcal{F}_k \right]$$

$$k = 0, 1, \ldots, t - 1.$$
Our goal is to prove that
\[ V(0; E_t) = \Theta(0). \]
This result follows by the application of Lemma 3 and Lemma 4.

**Lemma 3**
\[ V(t; E_t) = \Theta(t). \]

**Proof.** By using (5) and $\mathcal{F}_t$-measurability of $H_t$, we have
\[ V(t; E_t) = \mathbb{E}^R [E_t | \mathcal{F}_t] = \mathbb{E}^R [R_t H_t | \mathcal{F}_t] = \mathbb{E}^R [R_t | \mathcal{F}_t] \cdot H_t = A_t q_t^R H_t = \Theta(t). \]  

**Lemma 4**
\[ V(k + 1; E_t) = \Theta(k + 1) \implies V(k; E_t) = \Theta(k) \quad k = 0, 1, \ldots, t - 1. \]

**Proof.** By defining
\[ H_t^k = e^{-r(t-k)h} \mathbb{E}^Q \left[ \prod_{m=k}^{t-1} (1 - q_m^R) H_m q_m^R \bigg| \mathcal{F}_k \right] \]
we have
\[ \Theta(k) = A_k (t-k)h_{p_x+k} H_t^k. \]
From (5) and (1), we obtain
\[ V(k; E_t) = \mathbb{E}^R \left[ \mathbb{E} \left[ \pi_{k+1} V(k + 1; E_t) \bigg| \mathcal{F}_k \right] \bigg| \mathcal{F}_k \right]. \]
By using the induction hypothesis
\[ V(k + 1; E_t) = \Theta(k + 1) \]
we have
\[ V(k; E_t) = \mathbb{E}^R \left[ \mathbb{E} \left[ \pi_{k+1} \Theta(k + 1) \bigg| \mathcal{F}_k \right] \bigg| \mathcal{F}_k \right]. \]  
(16)

We have
\[ \mathbb{E} \left[ \pi_{k+1} \Theta(k + 1) \bigg| \mathcal{F}_k \right] = \mathbb{E} \left[ \chi_{k+1} A_{k+1} \bigg| \mathcal{F}_k \right] \cdot \mathbb{E} \left[ \varphi_{k+1} H_t^{k+1} \bigg| \mathcal{F}_k \right] \]
where
\[ \mathbb{E} \left[ \chi_{k+1} A_{k+1} \bigg| \mathcal{F}_k \right] = \mathbb{E}^D \left[ A_{k+1} \bigg| \mathcal{F}_k \right] = (A_k - R_k) \left( 1 - q_x H_t^{k+1} \right) \]
and
\[ E \left[ \varphi_{k,k+1} H_t^{k+1} \bigg| \mathcal{F}_k \right] = E^Q \left[ e^{-r_h} H_t^{k+1} \bigg| \mathcal{F}_k \right] = E^Q \left[ e^{-r_h} H_t^{k+1} \bigg| \mathcal{F}_k \right]. \]

Substituting in (16), we obtain
\[ V(k; E_t) = (t-k)hP_{x+(k+1)}h \ E^R \left[ (A_k - R_k) \left( 1 - q_{x+k}h \right) \bigg| \mathcal{F}_k \right] \cdot E^Q \left[ e^{-r_h} H_t^{k+1} \bigg| \mathcal{F}_k \right] \]
\[ = (t-k)hP_{x+(k+1)}h \ hP_{x+k}h A_k (1 - q_k^R) \ E^Q \left[ e^{-r_h} H_t^{k+1} \bigg| \mathcal{F}_k \right] \]
\[ = (t-k)hP_{x+k}h A_k E^Q \left[ e^{-r_h} (1 - q_k^R) H_t^{k+1} \bigg| \mathcal{F}_k \right]. \]

We can easily see that
\[ H_t^k = E^Q \left[ e^{-r_h} H_t^{k+1} (1 - q_{k+1}^R) \bigg| \mathcal{F}_k \right]. \]

We get
\[ V(k; E_t) = (t-k)hP_{x+k}h A_k H_t^k = (t-k)hP_{x+k}h A_k E^Q \left[ e^{-r(t-k)h} \prod_{m=k}^{t-1} (1 - q_m^R) H_t^R \bigg| \mathcal{F}_k \right] = \Theta(k). \]

This completes the proof. \( \square \)

By similar argument we can show (8).

**References**


